CMPS 211
Fall 2010 - 2011

Assignment 4 - Solution

Section 4.4

8- **procedure** *sum* (n: integer)
if ($n = 1$) $sum(n) := 1$
else $sum(n) := sum(n-1) + n$

10- **procedure** *largest* ($a_1, a_2, \ldots, a_n$: integers)
if ($n = 1$)
$largest(a_1, a_2, \ldots, a_n) := a_1$
else
$largest(a_1, a_2, \ldots, a_n) := \text{MAX}(largest(a_1, a_2, \ldots, a_{n-1}), a_n)$

{MAX returns the maximum between two integers}

24- **procedure** *exponent* (a: real, n: integer)
if ($n = 1$) $exponent(a, n) := a^2$
else $exponent(a, n) := [exponent(a, n-1)]^2$

30- **procedure** *iterativeA* (n: integer)
$a_0 := 1$
$a_1 := 2$
for $i := 2$ to n
$a_i = a_{i-1} \cdot a_{i-2}$

38- **procedure** *concatenation* (w: bit string, i: integer)
if ($i = 1$) $concatenation(w, i) := w$
else
$concatenation(w, i) := w \cdot concatentation(w, i-1)$

Section 4.5

2- Suppose that **T** is true.
If $x < 0$, then the program segment assigns the value of 0 to x.
If $x \geq 0$, then the program segment does not change the value of x.
In both cases, the resulting value of x is greater than or equal to 0, so the final assertion $x \geq 0$ is true.

4- Suppose that **T** is true, and given two input values x and y.
If $x < y$, then program segment assigns the value of *min* to x.
Otherwise, if $y \leq x$, then the program segment assigns the value of *min* to y.
All in all, the final assertion is true, since
$(x < y \wedge min = x)$ is true according to the first case, and
$(x \geq y \wedge min = y)$ is true according to the second case.

12- Initial assertion *I:* "*a* and *d* are positive integers".
Final assertion *F*: "*q* and *r* are integers such that $a = dq + r$ and $0 \leq r < d$".

The program segment initially assigns the value of *a* to *r* and 0 to *q*.
The *while* loop iterates until the condition "$r < d$" is satisfied, so the condition used is *C*: "$r \geq d$".

***Case* 1:** *I* is true and *C* is true.
The program segment enters the loop and executes the statements
$r := r - d$
$q := q + 1$
So, in each iteration, *r* is decremented by *d* and *q* is incremented by 1. In this case, *q* will maintain the counter of the number of times *r* was subtracted, i.e. the number of multiples of *d* removed from *r*. When the program segment exits the *while* loop, the value in *r* will be $r = a - d \cdot q$ (since *r* was initially equal to *a*). Also, the loop can exit only when $r < d$.
$\Rightarrow$ "*q* and *r* are integers such that $a = dq + r$ and $0 \leq r < d$" is true.
$\therefore I \wedge C \{S_1\} F$ is true.

***Case* 2:** *I* is true and *C* is NOT true.
The program segment does not enter the loop since $r < d$.
$\Rightarrow$ $r = a$ and $r < d$
$q = 0$
These values also result in the final assertion where $a = dq + r = 0 + r = r$.
$\Rightarrow$ "*q* and *r* are integers such that $a = dq + r$ and $0 \leq r < d$" is true.
$\therefore I \wedge \neg C \{S_2\} F$ is true.

All in all:

$$\frac{\begin{array}{l} I \wedge C \{S_1\} F \\ I \wedge \neg C \{S_2\} F \end{array}}{I \wedge \{\text{if } C \text{ then } S_1 \text{ else if } C \text{ then } S_2\} F}$$

Section 5.1

2- 27 floors and 37 offices on each floor.
Number of offices in the building = 27 × 37 = 999 offices.

14- Bit strings of length n, starting and ending with 1s = $1 \times 2^{n-2} \times 1 = 2^{n-2}$.
(First and last bit are 1s, so only option, and the remaining $n-2$ bits could be either 0 or 1, so two options).

20- Number of positive integers less than 1000 (0 included, 1000 not included):

a) divisible by 7 = $\left\lfloor \frac{1000}{7} \right\rfloor = 142$.

c) divisible by both 7 and 11 = divisible by 77 = $\left\lfloor \frac{1000}{77} \right\rfloor = 12$.

e) divisible by exactly one of 7 and 11 = divisible by 7 or by 11, but not both
$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - 2 \times \left\lfloor \frac{1000}{77} \right\rfloor = 142 + 90 - 2 \times 12 = 208$.
(include the count of the union of both sets but exclude the count of their intersection entirely, so must subtract it twice, once for each set count).

f) divisible by neither 7 nor 11 = not divisible by either 7 or 11 = all numbers – those divisible by either 7 or 11 =1000 – 220 = 780.

28- License plates made using either
three letters followed by three digits or four letters followed by two digits
= 26×26×26×10×10×10 + 26×26×26×26×10×10
= 17,576,000 + 45,697,600 = 63,273,600

30- Strings of eight English letters:
a) if letters can be repeated = 26^8 = 208,827,064,576
c) that start with X, if letters can be repeated = $1\times26^7 = 26^7$ = 8,031,810,176.
e) that start and end with X, if letters can be repeated = $1\times26^6\times1$ = 308,915,776.

40- Ways to arrange 6 people in a row from a group of 10 people if:
a) the bride must be in the picture ?
Number of ways = (1×9×8×7×6×5) × 6 = 90,720.
b) both the bride and the groom must be in the picture?
Number of ways = (1×1×8×7×6×5) × 6 × 5 = 50,400.

c) exactly one of the bride and the groom is in the picture?
Number of ways = (1×8×7×6×5×4) × 6 × 2 = 80, 640.

46- A student is either a computer science major or mathematics major or joint major. There are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors) and 7 joint majors.
So, the total number of students = 38 + 23 – 7 = 54 (using inclusion-exclusion).

Section 5.2

4- Bowl contains 10 red balls and 10 blue balls. Select balls at random.

a) Number of balls to select to be sure of having at least three balls of the same color = 5.

b) Number of balls to select to be sure of having at least three blue balls = 13.

14- a) Seven integers selected from the first 10 positive integers.
In the first 10 positive integers, there are 5 pairs of integers that sum up to 11 {(1,10), (2,9), (3,8), (4,7), (5,6)}.
If seven integers are selected from these 10, then by the pigeonhole principle, there must be two integers that belong to the same set as two other integers.
So, selecting seven integers guarantees that at least 2 pairs will sum up to 11.

b) No, selecting six integers only can guarantee at least 1 pair to sum up to 11, but not two. Take {1, 2, 3, 4, 5, 6}, then (5,6) is a pair with sum equal to 11, but there is no second pair.

16- In the given set, the following pairs add up to 16: {(1,15), (3,13), (5,11), (7,9)}.
To guarantee at least one pair of these numbers, we must select 5 integers out of the set.

32- In a computer network of 6 computers, each computer is directly connected to at least one of the other computers.
So, each computer is connected to a number of computers ranging from 1 to 5, i.e. there are at most 5 different possibilities for the number of computers any one can be connected to.
However, there are 6 computers in the network, so according to the pigeonhole principle, there must be at least 2 computers in the network connected to the same number of computers.

Section 5.3

8- For 5 runners, there are 5! = 120 different orders to finish the race.

12- Bit strings of length 12 containing:

a) exactly three 1s = C (12, 3) = $\frac{12!}{(12-3)!\times 3!} = \frac{12!}{9!\times 3!} = \frac{12\times 11\times 10}{3\times 2} = 220.$

c) at least three 1s = 2^{12} – [C (12,0) + C (12,1) + C (12,2)]
= 2^{12} – [1 + 12 + 66] = 4, 017.

18- Coin flipped 8 times. Possible outcomes:

a) in total = 2^{8} = 256.

c) containing at least three heads = 2^{8} – [C (8,0) + C (8,1) + C (8,2)]

$= 256 - [1 + 8 + \frac{8\times 7}{2}]$

= 256 – 37 = 219.

22- Permutations of the letters *ABCDEFGH* containing:

a) the string *ED* = 7 ! = 5,040.
(counting *ED* as a single block and applying permutation on the elements *A*, *B*, *C*, *ED*, *F*, *G*, *H*)

c) the strings *BA* and *FGH* = 5 ! = 120.
(counting *BA* and *FGH* as single blocks and applying permutation on the elements *BA*, *C*, *D*, *E*, *FGH*)

d) the strings *CAB* and *BED* = 4 ! = 24.
(counting *CAB* and *BED* as **one** single block since they must be joined at *B* and applying permutation on the elements *CABED*, *F*, *G*, *H*)

26- Thirteen people are on a softball team.

a) Number of ways to choose 10 players = C (13,10)

$= \frac{13!}{(13-10)!\times 10!} = \frac{13!}{3!\times 10!} = \frac{13\times 12\times 11}{3\times 2} = 286.$

b) Number of ways to assign the 10 positions = P (13,10) = $\frac{13!}{(13-10)!} = \frac{13!}{3!}$

= 13 × 12 × 11 × 10 × 9 × 8 × 7 × 6 × 5 × 4 = 1,037,836,800.

c) Three out of thirteen are women. Number of ways to choose 10 players if at

least one of these must be a woman = total number of ways to choose 10 players – number of ways to choose 10 men only (no women)
= C (13,10) – C (10,10) = 286 – 1 = 285.

Section 5.4

The Binomial Theorem: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

4- In $(x+y)^{13}$, the coefficient of x^5y^8 is obtained using the binomial theorem at n=13 and j=8; i.e. the coefficient is equal to

$$\binom{n}{j} = \binom{13}{8} = \frac{13!}{(13-8)! \times 8!} = \frac{13!}{5! \times 8!} = \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2} = 1287.$$

28- Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

a) Using a combinatorial argument:

We show that each side of the equation counts the number of ways to choose a subset of 2 elements from a set of $2n$ elements.

For the *left-hand* side, choose the 2 elements out of a set of $2n$ elements, which can be done in $\binom{2n}{2}$ ways.

For the *right-hand* side, choose the 2 elements out of two sets of n elements each. The possibilities to do that are:

- choose the 2 elements out of the first set and none out of the second
- choose 1 element out of the first set and 1 element out of the second
- choose the 2 elements out of the second set and none out of the first

$$\text{The number of ways} = \binom{n}{2}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{0}\binom{n}{2}$$

$$= \binom{n}{2} \times 1 + \frac{n!}{(n-1)! \times 1!} \frac{n!}{(n-1)! \times 1!} + 1 \times \binom{n}{2}$$

$$= 2\binom{n}{2} + n \times n = 2\binom{n}{2} + n^2$$

b) Using the formula:

$$\binom{2n}{2} = \frac{(2n)!}{(2n-2)! \times 2!} = \frac{2n \times (2n-1)}{2} = \frac{4n^2 - 2n}{2} = 2n^2 - n$$

$$2\binom{n}{2} + n^2 = 2\frac{n!}{(n-2)! \times 2!} + n^2 = 2\frac{n \times (n-1)}{2} + n^2 = n \times (n-1) + n^2 = 2n^2 - n$$

Therefore, $\binom{2n}{2} = 2\binom{n}{2} + n^2$.